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# Full soft subsystems, soft homomorphisms and soft closure operators over soft machines

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Abstract. Let $\mathfrak{S} = (P, M, F) \in \mathcal{SFM}(\mathcal{U})$ be a soft finite state machine, where $P$ is a finite nonempty set ( states ), $M$ is a finite nonempty set ( inputs ) and $(F, P \times M \times P) \in \mathcal{S}(\mathcal{U})$. In this paper, we define the mappings $S^{\upsilon} : \mathcal{P}(P) \longrightarrow \mathcal{P}(P), \forall \upsilon \subseteq \mathcal{U}$ and use them to introduce the concepts of soft $\upsilon$-successor and full soft subsystems, and their characterization is proposed. Soft homomorphisms and strong soft homomorphisms between soft finite state machines are defined. Finally, a soft closure operator is defined, which induces a soft topological space over $\mathcal{U}$. Characterization of full soft subsystems in terms of the soft topology is obtained.

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## 1. Introduction and basic definitions

Kleene [1] was the pioneer in introducing the concept of finite state machines (**fsm**), which have proven to be highly valuable in the design of computer circuits and various other fields. Building upon this foundational work, numerous researchers have made significant contributions to the advancement and generalization of **fsm** theory. For instance, the development of fuzzy finite state machines (**ffsm**) has been extensively studied in works such as [2, 3, 4], extending the classical concepts to accommodate uncertainty and vagueness. Additionally, the theory has been further expanded through the introduction of intuitionistic fuzzy finite state machines (**iffsm**) ), as explored in [5, 6], which incorporate an additional degree of hesitancy. More recently, bipolar fuzzy finite state machines (**bffsm**) have been proposed [7], offering a broader framework to model bipolarity and dual degrees of membership in complex systems. These developments reflect the rich and ongoing evolution of finite state machine theory to address increasingly sophisticated and nuanced applications.

Soft set is a new mathematical tool that Molodtsov [8] proposed to deal with uncertain situations without the problems that affect the usual theoretical methods. Soft set theory is a more general form of fuzzy set theory. There is a strong connection between the two theories [9, 10].

Algebraic structures like semigroups, groups and rings are applied to soft set theory [11, 12, 13, 14, 15]. Shabir and Naz [16] proposed the concept of soft topological spaces defined over an initial universe with a predetermined set of parameters. Recently, a new relationship between algebraic structures and soft topologies has been explored in [17]. In 2015, Hussain et al. [18] initiated the notion of soft finite state machins **sfsm**. They defined soft successors, soft subsystems of a **sfsm**. Also, they introduced soft submachines and discussed their direct product. Inspired by the observation that certain results identified by Hussain et al hold for fuzzy finite state machines (**ffsm**) but not for soft finite state machines (**sfsm**), we present this study to further investigate this distinction.

In this paper, given a soft machine $\mathfrak{S} = (P, M, F) \in \mathcal{SFM}(\mathcal{U})$, we define the mappings $S^{\upsilon} : \mathcal{P}(P) \longrightarrow \mathcal{P}(P)$, $\forall \upsilon \subseteq \mathcal{U}$ and use them to introduce the concepts of soft $\upsilon$-successor and full soft subsystems, and their characterization is proposed. Soft homomorphisms and strong soft homomorphisms between soft finite state machines are defined. Finally, a soft closure operator is defined, which induces a soft topological space over $\mathcal{U}$. Characterization of full soft subsystems in terms of the soft topology is obtain.

Suppose $\mathcal{U}$ is a universe set, $\mathcal{P}(\mathcal{U})$ is the set of all subsets of $\mathcal{U}$, and $\mathcal{E} \neq \phi$ is the set of all parameters.

**Definition 1.1** ([8])**.** Let $\mathcal{A} \subseteq \mathcal{E}$ and $F : \mathcal{A} \longrightarrow \mathcal{P}(\mathcal{U})$ be a set-valued mapping. Then the pair $(F, \mathcal{A})$ is called a *soft set* over $\mathcal{U}$.

We denote $\mathcal{S}(\mathcal{U})$ to the set of all soft sets $(F, \mathcal{A})$ over $\mathcal{U}$.

**Definition 1.2** ([8])**.** Let $(F, \mathcal{A}), (G, \mathcal{A}) \in \mathcal{S}(\mathcal{U})$ Then $(F, \mathcal{A})$ is a *subset* of $(G, \mathcal{A})$, denoted by $(F, \mathcal{A}) \sqsubseteq (G, \mathcal{A})$, if $F(a) \subseteq G(a)$ for all $a \in \mathcal{A}$ and $(F, \mathcal{A})$; $(G, \mathcal{A})$ are called *equal*, denoted by $(F, \mathcal{A}) = (G, \mathcal{A})$, if$F(a) = G(a)$ for all $a \in \mathcal{A}$.

**Definition 1.3** ([8])**.** Let $(F, \mathcal{A}), (G, \mathcal{A}) \in \mathcal{S}(\mathcal{U})$. Then the *union* $(F \sqcup G, \mathcal{A})$ and the *intersection* $(F \sqcap G, \mathcal{A})$ of $(F, \mathcal{A})$ and $(G, \mathcal{A})$ are soft sets defined by

$$(F \sqcup G)(a) = F(a) \cup G(a),$$
$$(F \sqcap G)(a) = F(a) \cap G(a),$$

respectively.

**Definition 1.4** ([19])**.** Let $\{(F_i, \mathcal{A}) : i \in I\} \subseteq \mathcal{S}(\mathcal{U})$, then

(i) $(\sqcup_i F_i, \mathcal{A}) \in \mathcal{S}(\mathcal{U})$ defined by $(\sqcup_i F_i)(a) = \cup_i F_i(a)\ \forall a \in \mathcal{A}$,
(ii) $(\sqcap_i F_i, \mathcal{A}) \in \mathcal{S}(\mathcal{U})$ defined by $(\sqcap_i F_i)(a) = \cap_i F_i(a)\ \forall a \in \mathcal{A}$,

**Definition 1.5** ([19])**.** Let $\mathcal{A}, \mathcal{B} \subseteq \mathcal{E}$ and $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ be a mapping. then

(i) the *image* of $(F, \mathcal{A}) \in \mathcal{S}(\mathcal{U})$ under the mapping $\phi$ is a soft set $(\phi(F), \mathcal{B}) \in \mathcal{S}(\mathcal{U})$ defined by

$$\phi(F)(b) = \begin{cases} \bigcup_{x \in \phi^{-1}(b)} F(x) & \text{if } \phi^{-1}(b) \neq \varnothing \\ \varnothing & \text{otherwise} \end{cases}$$

(ii) The *inverse image* of $(G, \mathcal{B}) \in \mathcal{S}(\mathcal{U})$ under the mapping $\phi$ is a soft set $(\phi^{-1}(G), \mathcal{A}) \in \mathcal{S}(\mathcal{U})$ such that $\phi^{-1}(G)(a) = G(\phi(a)), \forall a \in \mathcal{A}$.

**Definition 1.6** ([18])**.** A triple $\mathfrak{S} = (P, M, F)$ is called a *soft finite state machine* (briefly, **(sfsm),** where $P$ is a finite nonempty set (states), $M$ is a finite nonempty set (inputs) and $(F, P \times M \times P) \in \mathcal{S}(\mathcal{U})$.

Let $M^*$ denote the set of all words of elements of $M$ of finite length. The empty word is denoted by $\omega$, and $|m|$ denote the length of a word $m$. We shall denote by $\mathcal{SFM}(\mathcal{U})$ to the class of all soft finite state machines over $\mathcal{U}$.

**Definition 1.7** ([18])**.** Let $\mathfrak{S} = (P, M, F) \in \mathcal{SFM}(\mathcal{U})$. Define the *extension* $F^*$ : $P \times M^* \times P \longrightarrow \mathcal{P}(\mathcal{U})$ of $F$ as follows:

$$F^*(p, \omega, q) = \begin{cases} \mathcal{U} & \text{if } q = p \\ \varnothing & \text{if } q \neq p \end{cases}$$

$$F^*(p, m\sigma, q) = \bigcup_{r \in P} \{F^*(p, m, r) \cap F(r, \sigma, q)\}$$

for all $m \in M^*$, $p, q \in P$ and $\sigma \in M$.

**Lemma 1.8** ([18])**.** *Let* $\mathfrak{S} = (P, M, F) \in \mathcal{SFM}(\mathcal{U})$*. Then*

$$F^*(p, mn, q) = \bigcup_{r \in P} \{F^*(p, m, r) \cap F^*(r, n, q)\}$$

*for all* $m, n \in M^*$ *and* $p, q \in P$*.*

**Definition 1.9** ([18])**.** Let $\mathfrak{S} = (P, M, F) \in \mathcal{SFM}(\mathcal{U})$ and $p, q \in P$. Then $p$ is called a *soft immediate successor* of $q$, if there exists $\sigma \in M$ such that $F(q, \sigma, p) \neq \varnothing$. $p$ is called a *soft successor* of $q$, if there exists $m \in M^*$ such that $F^*(q, m, p) \neq \varnothing$. The set of all soft successors of q is denoted by $Su(q)$. For $X \subseteq P$, define

$$Su(X) = \cup \{Su(q) : q \in X\}.$$

**Proposition 1.10** ([18])**.** *Let* $\mathfrak{S} = (P, M, F) \in \mathcal{SFM}(\mathcal{U})$*. Then* $p \in Su(p)$ *for every* $p \in P$*.*

**Definition 1.11** ([18])**.** Let $\mathfrak{S} = (P, M, F) \in \mathcal{SFM}(\mathcal{U})$ and $(G, P) \in \mathcal{S}(\mathcal{U})$. Then $G$ is called a *soft subsystem* of $\mathfrak{S}$, if

$$G(p) \cap F(p, \sigma, q) \subseteq G(q)$$

for all $p, q \in P$ and $\sigma \in M$.

**Theorem 1.12** ([18])**.** *Let* $\mathfrak{S} = (P, M, F) \in \mathcal{SFM}(\mathcal{U})$ *and* $(G, P) \in \mathcal{S}(\mathcal{U})$*. Then* $G$ *is a soft subsystem of* $\mathfrak{S}$ *if and only if*

$$G(p) \cap F^*(p, m, q) \subseteq G(q)$$

*for all* $p, q \in P$ *and* $m \in M^*$*.*

## 2. Full soft subsystems

**Definition 2.1** ([20])**.** For a soft set $(F, P) \in \mathcal{S}(\mathcal{U})$ and $\upsilon \in \mathcal{P}(\mathcal{U})$. Define the $\upsilon$*-support* of $F$ by the set

$$F^\upsilon = \{x \in P : \upsilon \subseteq F(x)\}.$$

Next, we use the concept of $\upsilon-$support to introduce the definition of $\upsilon-$successors, which contains Definition 1.9.

**Definition 2.2.** Let $\mathfrak{S} = (P, M, F) \in \mathcal{SFM}(\mathcal{U})$ and $p, q \in P$. Then $p$ is called

(i) an *immediate* $\upsilon$*-successor* of $q$, if there exists $\sigma \in M$ such that $F(q, \sigma, p) \supseteq \upsilon$.
(ii) a $\upsilon$*-successor* of $q$, if there exists $m \in M^*$ such that $F^*(q, m, p) \supseteq \upsilon$.

The set of all $\upsilon-$successors of $q$ is denoted by $S^\upsilon(q)$. For $X \subseteq P$, define

$$S^\upsilon(X) = \cup\{S^\upsilon(q) : q \in X\}.$$

This is a mapping from the power set $\mathcal{P}(P)$ into itself.

**Remark 2.3.** For every $\varnothing \neq \upsilon \in \mathcal{P}(\mathcal{U})$, $p \in S^\upsilon(q) \Rightarrow p \in Su(q)$, by Definition 1.9. The converse of this fact does not hold in general.

For all $\upsilon \in \mathcal{P}(\mathcal{U})$, using the mapping $S^\upsilon : \mathcal{P}(P) \longrightarrow \mathcal{P}(P)$, we define full soft subsystems of a **sfsm** over $\mathcal{U}$ as follows:

**Definition 2.4.** Let $\mathfrak{S} = (P, M, F) \in \mathcal{SFM}(\mathcal{U})$. and $(G, P) \in \mathcal{S}(\mathcal{U})$. Then $G$ is called a *soft* $\upsilon$*-subsystem* of $\mathfrak{S}$, if

$$S^\upsilon(G^\upsilon) \subset G^\upsilon.$$

$G$ is called a *full soft subsystem* of $\mathfrak{S}$, if it is a soft $\upsilon$-subsystem of $\mathfrak{S}$ for every $\upsilon \in \mathcal{P}(\mathcal{U})$.

**Example 2.5.** Let $\mathfrak{S} = (P, M, F) \in \mathcal{SFM}(\mathcal{U})$, where $P = \{p, q\}$ and $M = \{\sigma\}$. Consider $\mathcal{U} = \{1, 2, 3, 4\}$. We define $F : P \times M \times P \longrightarrow \mathcal{P}(\mathcal{U})$ by $F(r, \sigma, t) = \upsilon \in \mathcal{P}(\mathcal{U})$ for all $r, t \in P$.

- If $(L, P) \in \mathcal{S}(\mathcal{U})$ such that $L(p) = L(q) = \upsilon$. It is clear that $S^\upsilon(L^\upsilon) = P = L^\upsilon$ for all $\upsilon \in \mathcal{P}(\mathcal{U})$. This means that $L$ is a full soft subsystem of $\mathfrak{S}$.
- If $(G, P) \in \mathcal{S}(\mathcal{U})$ such that $G(p) = \{1, 3\} = \upsilon$ and $G(q) = \{1, 2, 3\} \supseteq \upsilon$. It is clear that $G^\upsilon = P$. Then we have $S^\upsilon(G^\upsilon) = P = G^\upsilon$ which implies that $G$ is a soft $\upsilon-$subsystem of $\mathfrak{S}$.
- If $(H, P) \in \mathcal{S}(\mathcal{U})$ such that $G(p) = \{1, 3\} = \upsilon$ and $G(q) = \{1, 4\}$. Then $H^\upsilon = \{p\}$ and $S^\upsilon(H^\upsilon) = P$ is not a subset of $H^\upsilon$. Thus $H$ is not a soft $\upsilon-$subsystem of $\mathfrak{S}$.

The following theorem shows that Definition 2.4 and Definition 1.11 are equivalent.

**Theorem 2.6.** *Let* $\mathfrak{S} = (P, M, F) \in \mathcal{SFM}(\mathcal{U})$, *and* $(G, P) \in \mathcal{S}(\mathcal{U})$. *Then* $\mathfrak{N} = (P, F, M, G)$ *is a full soft subsystem of* $\mathfrak{S}$ *if and only if*

$$G(p) \supseteq G(q) \cap F^*(q, m, p)$$

*for all* $p, q \in P$ *and* $m \in M^*$.

*Proof.* ($\Rightarrow$) Suppose $\mathfrak{N} = (P, F, M, G)$ is a full soft subsystem of $\mathfrak{S}$. Let $p \in P$ such that

$$G(q) \cap F^*(q, m, p) \supset G(p) \tag{2.1}$$

for some $q \in P$ and $m \in M^*$. Choose $\upsilon = G(q) \cap F^*(q, m, p)$. If $\upsilon = \varnothing$, the result holds. Suppose $\upsilon \neq \varnothing$. Then $\upsilon \subseteq G(q)$ and $\upsilon \subseteq F^*(q, m, p)$. Since $\mathfrak{N}$ is a full soft subsystem of $\mathfrak{S}$, $p \in S^\upsilon(q) \subseteq S^\upsilon(G^\upsilon) \subseteq G^\upsilon$ contradicts with (2.1).

($\Leftarrow$) Suppose $q \in S^\upsilon(G^\upsilon)$ for every $\upsilon \in \mathcal{P}(\mathcal{U})$. Then there exist $p \in G^\upsilon$ and $m \in M^*$ such that $F^*(p, m, q) \supseteq \upsilon$. Thus we have

$$\upsilon \subseteq G(p) \cap F^*(p, m, q) \subseteq G(q)$$

which implies that $q \in G^\upsilon$. The proof completes. □

**Definition 2.7.** Let $\mathfrak{S} = (P, M, F) \in \mathcal{SFM}(\mathcal{U})$ and $(G, P) \in \mathcal{S}(\mathcal{U})$. For all $m \in M^*$, the soft set $(G_m, P) \in \mathcal{S}(\mathcal{U})$ is defined by

$$G_m(q) = \bigcup_{p \in P} \{G(p) \cap F^*(p, m, q)\}.$$

**Theorem 2.8.** *Let $\mathfrak{S} = (P, M, F) \in \mathcal{SFM}(\mathcal{U})$, and $(G, P) \in \mathcal{S}(\mathcal{U})$. Then*

$$G \text{ is a full soft subsystem of } \mathfrak{S} \iff (G_m, P) \sqsubseteq (G, P), \forall m \in M^*.$$

*Proof.* Suppose $G$ is a full soft subsystem of $\mathfrak{S}$. Then for all $q \in P, m \in M^*$,

$$G_m(q) = \bigcup\nolimits_{p \in P} \{G(p) \cap F^*(p, m, q)\} \subseteq G(q).$$

Thus $(G_m, P) \sqsubseteq (G, P)$.

Conversely, suppose $G_m(q) \subseteq G(q) \forall m \in M^*, q \in P$. Then we obtain

$$G(q) \supseteq G_m(q) = \bigcup\nolimits_{p \in P} \{G(p) \cap F^*(p, m, q)\} \supseteq G(p) \cap F^*(p, m, q).$$

Thus by Theorem 2.6, $G$ is a full soft subsystem of $\mathfrak{S}$. □

## 3. Soft homomorphisms

**Definition 3.1.** Let $\mathfrak{S}_i = (P_i, M_i, F_i) \in \mathcal{SFM}(\mathcal{U}), i = 1, 2$, and $\phi : P_1 \longrightarrow P_2, \psi : M_1 \longrightarrow M_2$ be functions. Then the pair $(\phi, \psi) : \mathfrak{S}_1 \longrightarrow \mathfrak{S}_2$ is called

(i) a *soft homomorphism*, if

$$F_2(\phi(q), \psi(\sigma), \phi(p)) \supseteq F_1(q, \sigma, p)$$

for every $p, q \in P_1, \sigma \in M_1$,

(ii) a *strong soft homomorphism*, if

$$F_2(\phi(q), \psi(\sigma), \phi(p)) = \bigcup\nolimits_{r \in P_1} \{F_1(q, \sigma, r),\ \phi(r) = \phi(p)\}$$

for every $p, q \in P_1, \sigma \in M_1$.

A soft homomorphism (strong soft homomorphism) $(\phi, \psi) : \mathfrak{S}_1 \longrightarrow \mathfrak{S}_2$ is called a *soft isomorphism* (*strong soft isomorphism*), if $\phi$ and $\psi$ are bijective (1-1 and onto) mappings.

**Example 3.2.** Let $\mathfrak{S}_i = (P_i, M_i, F_i) \in \mathcal{SFM}(\mathcal{U}), i = 1, 2$ such that $P_1 = \{p_1, p_2, p_3\}, M_1 = M_2 = \{\alpha, \beta\}$ and $P_2 = \{q_1, q_2, q_3\}$. consider $F_1 : P_1 \times M_1 \times P_1 \longrightarrow \mathcal{P}(\mathcal{U})$ and $F_2 : P_2 \times M_2 \times P_2 \longrightarrow \mathcal{P}(\mathcal{U})$ are soft sets over $\mathcal{U} = \{1, 2, 3, 4\}$ defined by

$$F_1(p_1, \alpha, p_1) = \{1, 3\} = F_2(q_1, \alpha, q_1), \quad F_1(p_1, \beta, p_2) = \{1, 2, 3\} = F_2(q_1, \beta, q_2),$$

$$F_1(p_2, \alpha, p_1) = \{2, 4\} = F_2(q_2, \alpha, q_1), \quad F_1(p_3, \alpha, p_3) = \{1, 3\} = F_2(q_3, \alpha, q_1),$$

$$F_1(p_2, \beta, p_3) = \{2, 4\} = F_2(q_2, \beta, q_1), \quad F_1(p_3, \beta, p_2) = \{1, 2, 3\} = F_2(q_3, \beta, q_2).$$

For all other triples, $F_1 = F_2 = \varnothing$. Define the pair $(\phi, \psi) : \mathfrak{S}_1 \longrightarrow \mathfrak{S}_2$ as follows: $\phi(p_1) = \phi(p_3) = q_1, \phi(p_2) = q_2$ and $\psi(\alpha) = \alpha, \psi(\beta) = \beta$. We conclude that $(\phi, \psi)$ is a strong soft homomorphism. Indeed, we have

$$\begin{aligned}
F_2(\phi(p_1), \psi(\alpha), \phi(p_1)) =& F_2(q_1, \alpha, q_1) = \{1, 3\} \\
=& F_1(p_1, \alpha, p_1) \cup F_1(p_1, \alpha, p_3), \\
F_2(\phi(p_1), \psi(\beta), \phi(p_2)) =& F_2(q_1, \beta, q_2) = \{1, 2, 3\} \\
=& F_1(p_1, \beta, p_2), \\
F_2(\phi(p_2), \psi(\alpha), \alpha(p_1)) =& F_2(q_2, \alpha, q_1) = \{2, 4\} \\
=& F_1(p_2, \alpha, p_1) \cup F_1(p_2, \alpha, p_3), \\
F_2(\phi(p_2), \psi(\beta), \phi(p_3)) =& F_2(q_2, \beta, q_1) = \{2, 4\} \\
=& F_1(p_2, \beta, p_3) \cup F_1(p_2, \beta, p_1), \\
F_2(\phi(p_3), \psi(\alpha), \phi(p_3)) =& F_2(q_1, \alpha, q_1) = \{1, 3\} \\
=& F_1(p_3, \alpha, p_3) \cup F_1(p_3, \alpha, p_1), \\
F_2(\phi(p_3), \psi(\beta), \phi(p_2)) =& F_2(q_1, \beta, q_2) = \{1, 2, 3\} \\
=& F_1(p_3, \beta, p_2).
\end{aligned}$$

**Remark 3.3.** In Definition 3.1, we simply write $f : \mathfrak{S}_1 \longrightarrow \mathfrak{S}_2$, if $M_1 = M_2$ and $g$ is the identity function. If $(f, g) : \mathfrak{S}_1 \longrightarrow \mathfrak{S}_2$ is a strong soft homomorphism with injective $f$, then

$$F_2(f(p), g(\sigma), f(q)) = F_1(p, \sigma, q)$$

for every $p, q \in Q_1, \sigma \in M_1$.

**Theorem 3.4.** *Let $\mathfrak{S}_i = (P_i, M_i, F_i, ) \in \mathcal{SFM}(\mathcal{U})$, $i = 1, 2$ and $(\phi, \psi) : \mathfrak{S}_1 \longrightarrow \mathfrak{S}_2$ be a strong soft homomorphism with onto functions. If $G$ is a soft subsystem of $\mathfrak{S}_1$, then $\phi(G)$ is a soft subsystem of $\mathfrak{S}_2$.*

*Proof.* Suppose $G$ is a soft subsystem of $\mathfrak{S}_1$. Let $s, t \in P_2$ and $\sigma \in M_2$. Then

$$\begin{aligned}
\phi(G)(s) \cap F_2(s, \sigma, t) &= (\cup\{G(r) : r \in P_1, \phi(r) = s\}) \cap F_2(s, \sigma, t) \\
&= \cup\{G(r) \cap F_2(s, \sigma, t) : r \in P_1, \phi(r) = s\}.
\end{aligned}$$

Let $p, q \in P_1, \alpha \in M_1$. Since the pair $(\phi, \psi)$ onto such that $\phi(p) = s, \phi(q) = t, \psi(\alpha) = \sigma$,

$$\begin{aligned} G(r) \cap F_2(s, \sigma, t) =& G(r) \cap F_2(\phi(p), \psi(\alpha), \phi(q)) \\ &= G(r) \cap (\cup\{F_1(p, \alpha, \ell) : \ell \in P_1, \phi(\ell) = \phi(q) = t\}) \\ &= \cup\{G(\ell) \cap F_1(p, \alpha, \ell) : \ell \in P_1, \phi(\ell) = \phi(q) = t\} \\ &\subseteq \cup\{G(\ell) : \ell \in P_1, \phi(\ell) = t\} = \phi(G)(t). \end{aligned}$$

Thus $\phi(G)(s) \cap F_2(s, \alpha, t) \subseteq \phi(G)(t)$. So Definition 1.11 implies that $\phi(G)$ is a soft subsystem of $\mathfrak{S}_2$. □

The following example illustrates that the theorem may not apply if the pair $(\phi, \psi)$ is not onto.

**Example 3.5.** Let $\mathfrak{S} = (P, M, F) \in \mathcal{SFM}(\mathcal{U})$ where $P = \{p, q\}$ and $M = \{\sigma\}$. Define $F : P \times M \times P \longrightarrow \mathcal{P}(\mathcal{U})$ by

$$F(p, \sigma, p) = F(q, \sigma, q) = U,\ F(p, \sigma, q) = \varnothing \neq A \subseteq U,\ F(q, \sigma, p) = U \setminus A$$

Let the non-surjective mapping $\phi : \mathfrak{S} \longrightarrow \mathfrak{S}$ be defined as: $\phi(p) = \phi(q) = p$. Since

$$F(p, \sigma, p) = F(\phi(q), \sigma, \phi(p)) = F(q, \sigma, p) \cup F(q, \sigma, q) = U,$$

$$F(p, \sigma, p) = F(\phi(p), \sigma, \phi(q)) = F(p, \sigma, p) \cup F(p, \sigma, q) = U,$$

$$F(p, \sigma, p) = F(\phi(q), \sigma, \phi(q)) = F(q, \sigma, q) \cup F(q, \sigma, p) = U,$$

$$F(p, \sigma, p) = F(\phi(q), \sigma, \phi(q)) = F(q, \sigma, p) \cup F(q, \sigma, q) = U,$$

$\phi$ is a strong soft homomorphism. Consider the soft set $G : P \longrightarrow \mathcal{P}(\mathcal{U})$ defined by $G(p) = G(q) = \varnothing \neq B \subset A \subseteq U$. Then $G$ is a soft subsystem of $\mathfrak{S}$ because

$$G(r) \supseteq G(s) \cap F(s, \sigma, r)\ \forall r, s \in P,$$

while, we have $\varnothing = \phi(G)(q) \not\supseteq G(p) \cap F(p, \sigma, q) = B$. Thus $\phi(G)$ is not a soft subsystem of $\mathfrak{S}$.

**Theorem 3.6.** *Let $\mathfrak{S}_i = (P_i, M_i, F_i,) \in \mathcal{SFM}(\mathcal{U})$, $i = 1, 2$ and $(\phi, \psi) : \mathfrak{S}_1 \longrightarrow \mathfrak{S}_2$ be a soft homomorphism. If $H$ is a full soft subsystem of $\mathfrak{S}_2$, then $\phi^{-1}(H)$ is a full soft subsystem of $\mathfrak{S}_1$.*

*Proof.* By Definition 2.4, the proof is done when the following condition is met.

$$S^{\upsilon}(\phi^{-1}(H)^{\upsilon}) \subseteq \phi^{-1}(H)^{\upsilon}\ \forall \upsilon \in \mathcal{P}(\mathcal{U}).$$

Let $p \in S^{\upsilon}(\phi^{-1}(H)^{\upsilon}) = S^{\upsilon}(\phi^{-1}(H^{\upsilon}))$. Then there exist $q \in \phi^{-1}(H^{\upsilon})$, $\sigma \in M_1$ such that $F_1(q, \sigma, p) \supseteq \upsilon$ and $\phi(q) \in H^{\upsilon}$. Since $(\phi, \psi)$ is a soft homomorphism, $F_2(\phi(q), \psi(\sigma), \phi(p)) \supseteq F_1(q, \sigma, p) \supseteq \upsilon$. Because $H$ is a full soft subsystem, we have

$$\phi(p) \in S^{\upsilon}(H^{\upsilon}) \subseteq H^{\upsilon}.$$

Thus $\phi(p) \in H^{\upsilon}$. So $p \in \phi^{-1}(H^{\upsilon}) = \phi^{-1}(H)^{\upsilon}$. Hence $\phi^{-1}(H)$ is a full soft subsystem of $\mathfrak{S}_1$. □

## 4. Soft topologies Vs soft finite state machine

Throughout this section, the class of all soft sets $(F, P)$ over $\mathcal{U}$ parameterized by $P$ is denoted by $\mathcal{S}(\mathcal{U}, P)$. Also, we write $F \in \mathcal{S}(\mathcal{U}, P)$ instead $(F, P) \in \mathcal{S}(\mathcal{U})$. Soft set $\Phi, \mathbf{U} \in \mathcal{S}(\mathcal{U}, P)$ such that $\Phi(p) = \varnothing$, $\mathbf{U}(p) = \mathcal{U} \quad \forall p \in P$, are called the *empty soft set* and the *whole soft set*, respectively.

**Definition 4.1** ([16])**.** A collection $T$ of $\mathcal{S}(\mathcal{U}, P)$ is called a *soft topology* over $\mathcal{U}$, if $T$ satisfies the following:

(ST1) $\Phi, \mathbf{U} \in T$,
(ST2) $F, G \in T \Rightarrow F \sqcap G \in T$,
(ST3) If $\{F_i, i \in I\} \subseteq T$, then $\bigsqcup_i F_i \in T$.

The triple $(\mathcal{U}, P, T)$ is called a *soft topological space* over $\mathcal{U}$. Every element $G \in T$ is called a $T$*-soft open set* and the complement of $G$ is called a $T$*-soft closed set*.

**Definition 4.2** ([21])**.** A mapping $cl : \mathcal{S}(\mathcal{U}, P) \longrightarrow \mathcal{S}(\mathcal{U}, P)$ is called a *soft closure operator* on $\mathcal{U}$, if it has the following properties for every $F, G \in \mathcal{S}(\mathcal{U}, P)$,

(C1) $cl(\Phi) = \Phi$,
(C2) $F \sqsubseteq cl(F)$,
(C3) $cl(F \sqcup G) = cl(F) \sqcup cl(G)$,
(C4) $cl(F) = cl(cl(F))$.

Theorem 1 in [21] showed that the soft clousre operator $cl$ induces a soft topology $T$ over $\mathcal{U}$ such that $cl(F)$ is the soft closure of $F$ according to the topology $T$.

Let $\mathfrak{S} = (P, M, F) \in \mathcal{SFM}(\mathcal{U})$ be a soft finite state machine and $G \in \mathcal{S}(\mathcal{U}, P)$. Define a soft set $\mathfrak{c}(F) \in \mathcal{S}(\mathcal{U}, P)$ by

$$\mathfrak{c}(G)(q) = \bigcup\nolimits_{p \in P} \{\bigcup\nolimits_{m \in M^*} \{G(p) \cap F^*(p, m, q)\}\} \quad \forall q \in P.$$

The following result states how the mapping $\mathfrak{c} : \mathcal{S}(\mathcal{U}, P) \longrightarrow \mathcal{S}(\mathcal{U}, P)$ derives a soft topology over $\mathcal{U}$.

**Theorem 4.3.** *The mapping $\mathfrak{c} : \mathcal{S}(\mathcal{U}, P) \longrightarrow \mathcal{S}(\mathcal{U}, P)$ is a soft closure operator. $\mathfrak{c}$ induces a soft topology $T$ over the set $\mathcal{U}$ having the property that $\mathfrak{c}(G)$ is the $T-$soft closure of $G \in \mathcal{S}(\mathcal{U}, P)$.*

*Proof.* To show that $\mathfrak{c}$ is a soft closure operator over $\mathcal{U}$, we need to verify the axioms of Definition 4.2.
**(C1):** It is straightforward.
**(C2):** For all $q \in P$, we have

$$\begin{aligned}\mathfrak{c}(G)(q) &= \bigcup\nolimits_{p \in P} \{\bigcup\nolimits_{m \in M^*} \{G(p) \cap F^*(p, m, q)\}\} \\ &\supseteq G(q) \cap F^*(q, \omega, q) = G(q).\end{aligned}$$

**(C3):** For all $G, H \in \mathcal{S}(\mathcal{U}, P)$, we have

$$\begin{aligned}
\mathfrak{c}(G \sqcup H)(q) &= \bigcup_{p\in P}\{\bigcup_{m\in M^*}\{G \sqcup H(p) \cap F^*(p,m,q)\}\} \\
&= \bigcup_{p\in P}\{\bigcup_{m\in M^*}\{G(p) \cup H(p) \cap F^*(p,m,q)\}\} \\
&= \bigcup_{p\in P}\{\bigcup_{m\in M^*}\{[G(p) \cap F^*(p,m,q)] \cup [H(p) \cap F^*(p,m,q)]\}\} \\
&= \bigcup_{p\in P}\{\bigcup_{m\in M^*}\{[G(p) \cap F^*(p,m,q)\}\} \cup \bigcup_{p\in P}\{\bigcup_{m\in M^*}\{H(p) \cap F^*(p,m,q)\}\} \\
&= \mathfrak{c}(G)(q) \cup \mathfrak{c}(H)(q) = \mathfrak{c}(G) \sqcup \mathfrak{c}(H)(q).
\end{aligned}$$

**(C4):** For all $q \in P$,

$$\begin{aligned}
\mathfrak{c}(\mathfrak{c}(G))(q) &= \bigcup_{p\in P}\{\bigcup_{m\in M^*}\{\mathfrak{c}(G)(p) \cap F^*(p,m,q)\}\} \\
&= \bigcup_{p\in P}\{\bigcup_{m\in M^*}\{[\bigcup_{r\in P}\{\bigcup_{n\in M^*}\{G(r) \cap F^*(r,n,p)\}\}] \cap F^*(p,m,q)\}\} \\
&= \bigcup_{p\in P}\{\bigcup_{m\in M^*}\{\bigcup_{r\in P}\{\bigcup_{n\in M^*}\{G(r) \cap F^*(r,n,p) \cap F^*(p,m,q)\}\}\}\} \\
&= \bigcup_{r\in P}\{\bigcup_{m\in M^*}\{\bigcup_{n\in M^*}\{\bigcup_{p\in P}\{G(r) \cap F^*(r,n,p) \cap F^*(p,m,q)\}\}\}\} \\
&= \bigcup_{r\in P}\{\bigcup_{m\in M^*}\{\bigcup_{n\in M^*}\{G(r) \cap \{\bigcup_{p\in P} F^*(r,n,p) \cap F^*(p,m,q)\}\}\}\} \\
&= \bigcup_{r\in P}\{\bigcup_{m\in M^*}\{\bigcup_{n\in M^*}\{G(r) \cap F^*(r,nm,q)\}\}\} \\
&\subseteq \bigcup_{r\in P}\{\bigcup_{z\in M^*}\{G(r) \cap F^*(r,z,q)\}\} = \mathfrak{c}(G)(q).
\end{aligned}$$

By the monotonicity of $\mathfrak{c}$ (easy to check), we verified that $\mathfrak{c}(G) = \mathfrak{c}(\mathfrak{c}(G))$. Then $\mathfrak{c}$ is a soft closure operator over $\mathcal{U}$. □

**Proposition 4.4.** *$G \in \mathcal{S}(\mathcal{U}, P)$ is a full soft subsystem of $\mathfrak{S} = (P, M, F) \in \mathcal{SFM}(\mathcal{U})$ if and only if $\mathfrak{c}(G) = G$.*

*Proof.* Let $G$ be a full soft subsystem of $\mathfrak{S}$. Beside axiom **(C2)**, It is enough to demonstrate thatt $\mathfrak{c}(G)(q) \subseteq G(q)\ \forall q \in P$. Let $q \in P$. Then as $G$ is a full soft subsystem of $\mathfrak{S}$, $G(p) \cap F^*(p,m,q) \subseteq G(q)\ \forall p \in P$. Consequently,

$$\mathfrak{c}(G)(q) = \bigcup_{p\in P}\{\bigcup_{m\in M^*}\{G(p) \cap F^*(p,m,q)\}\} \subseteq G(q).$$

Thus $\mathfrak{c}(G) = G$. Conversely for all $q \in P$, we have

$$G(q) = \mathfrak{c}(G)(q) = \bigcup_{p\in P}\{\bigcup_{m\in M^*}\{G(p) \cap F^*(p,m,q)\}\} \supseteq G(p) \cap F^*(p,m,q).$$

So by Theorem 2.6, $G$ is a full soft subsystem of $\mathfrak{S}$. □

**Theorem 4.5.** *Let $\mathfrak{S}_1 = (P_1, M, F_1) \in \mathcal{SFM}(\mathcal{U})$ with associated soft topology $T_1$ and $f : \mathfrak{S} \longrightarrow \mathfrak{S}_1$ be homomorphism. Then $f : (\mathcal{U}, P, T) \longrightarrow (\mathcal{U}, P_1, T_1))$ is a soft continuous.*

*Proof.* Let $G$ be a $T_1$-soft closed set. by Theorem 4.3, $\mathfrak{c}(G) = G$. Now $\forall q \in P$

$$\begin{aligned}\mathfrak{c}(f^{-1}(G))(q) &= \bigcup_{p\in P}\{\bigcup_{m\in M^*}\{f^{-1}(G)(p) \cap F^*(p,m,q)\}\} \\ &= \bigcup_{p\in P}\{\bigcup_{m\in M^*}\{G(f(p)) \cap F^*(p,m,q)\}\} \\ &\subseteq \bigcup_{p\in P}\{\bigcup_{m\in M^*}\{G(f(p)) \cap F_1^*(f(p),m,f(q))\}\} \\ &\subseteq \bigcup_{p_1\in P_1}\{\bigcup_{m\in M^*}\{G(p_1) \cap F_1^*(p_1,m,f(q))\}\} \\ &= \mathfrak{c}(G)(f(q)) = G(f(q)) = f^{-1}(G)(q).\end{aligned}$$

Axiom(C2) implies that $\mathfrak{c}(f^{-1}(G)) = f^{-1}(G)$. Then $f^{-1}(G)$ is a $T$-soft closed set. Thus $f : (\mathcal{U}, P, T) \longrightarrow (\mathcal{U}, P_1, T_1))$ is a soft continuous. □

## 5. Conclusion

This paper introduces mappings for soft finite state machines to define soft $\upsilon$-successors and full soft subsystems, with their characterizations. We also defined soft homomorphisms and a soft closure operator that induces a soft topological space over $\mathcal{U}$. These results provide a foundational framework for analyzing soft subsystems within a soft topological setting. In [22], Al-Shami proposed new types of soft separation axioms to initiate various families of soft spaces and discussed their application in optimal choice scenarios using these topological concepts. Future research may explore decision-making problems that arise within the soft topology induced by soft finite state machines.

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